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THE CHARACTERIZATION OF NULL GENERALIZED HELICES IN 5-DIMENSIONAL LORENTZIAN SPACE

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Abstract

In this paper, we study null generalized helices by describing in view of harmonic curvatures to a null Frenet curve of osculating order 5 in 5-dimensional Lorentzian space by using the Frenet frame consisting of two null and three space-like vectors from [3].

1. Introduction

Let $x = (x_1, x_2, x_3, x_4, x_5)$ and $y = (y_1, y_2, y_3, y_4, y_5)$ be two nonzero vectors in Minkowski 5-space $\mathbb{R}_1^5$. We denote $\mathbb{R}_1^5$ shortly by $\mathbb{L}^5$. For $x, y \in \mathbb{L}^5$,

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{5} x_i y_i,$$

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is called *Lorentzian inner product*. The couple $\{\mathbb{R}_1^5, \langle,\rangle\}$ is called *Lorentzian space* and briefly denoted by $\mathbb{L}^5$. Then a vector v of $\mathbb{L}^5$ is called

(i) time-like if $\langle v, v\rangle < 0$,

(ii) space-like if $\langle v, v\rangle > 0$ or $v = 0$,

(iii) null (or light-like) vector if $\langle v, v\rangle = 0$, $v \neq 0$.

An arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{L}^5$ can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(t)$ are, respectively, space-like, time-like or null [6].

2. Basic Definitions

Definition 1 [6]**.** On a semi-Riemannian manifold $M \subset \mathbb{L}^5$, there is a unique connection ∇ such that

$$[V, W] = \nabla_V W - \nabla_W V,$$

and

$$X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle,$$

for all $X, V, W \in \chi(\mathbb{L}^5)$. ∇ is called the *Levi-Civita connection* of $\mathbb{L}^5$.

Definition 2. Let $\alpha : I \longrightarrow \mathbb{L}^5$ be a null curve in $\mathbb{L}^5$. The curve α is called *Frenet curve of osculating order* 5, if its 5-th order derivatives $\alpha'(t), \alpha''(t), \alpha'''(t), \alpha^{iv}(t), \alpha^{v}(t)$ are linearly independent and $\alpha'(t), \alpha''(t)$, $\alpha'''(t), \alpha^{iv}(t), \alpha^{v}(t), \alpha^{vi}(t)$ are no longer linearly independent for all $t \in I$. For each null Frenet curve of osculating order 5, one can associate an orthonormal 5-frame $\{T, N, W_1, W_2, W_3\}$ along α (such that $\alpha'(t) = T$) called the *Frenet frame* and functions $\{k_1, k_2, k_3, k_4, k_5\}$ called the *Frenet curvatures*. Thus from [3], the Frenet equations of a null curve in a 5-dimensional Lorentz manifold are written down as follows:

$$\begin{cases} \nabla_T T = hT + k_1 W_1, \\ \nabla_T N = -hN + k_2 W_1 + k_3 W_2, \\ \nabla_T W_1 = -k_2 T - k_1 N + k_4 W_2 + k_5 W_3, \\ \nabla_T W_2 = -k_3 T - k_4 W_1, \\ \nabla_T W_3 = -k_5 W_1, \end{cases}$$

where ∇ is the Levi-Civita connection of $\mathbb{L}^5$; h and $\{k_1, k_2, k_3, k_4, k_5\}$ are differential functions; T and N are null vectors; W_1, W_2, and W_3 are space-like vectors. In these equations by changing a suitable parameter t, we may take $h = 0$ and other equations stay unchanged. This parameter is called distinguished parameter of the curve [3]. That is,

$$\begin{cases} \nabla_T T = k_1 W_1, \\ \nabla_T N = k_2 W_1 + k_3 W_2, \\ \nabla_T W_1 = -k_2 T - k_1 N + k_4 W_2 + k_5 W_3, \\ \nabla_T W_2 = -k_5 W_1. \end{cases} \tag{1}$$

From [3] again, since T and N are null vectors, $W_i, 1 \leq i \leq 3,$ are space-like vectors, then we have

$$\begin{cases} \langle T, T \rangle = 0, \langle N, N \rangle = 0, \langle T, N \rangle = 1, \langle T, W_i \rangle = 0, \langle N, W_i \rangle = 0, \\ \langle W_i, W_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{for } i, j = 1, 2, 3. \end{cases} \tag{2}$$

Definition 3. If a null curve $\alpha : I \longrightarrow \mathbb{L}^5$ is a null Frenet curve of osculating order 5 and Frenet curvatures $k_i, 1 \leq i \leq 5$ are nonzero constant, then α is called a *null W-curve of rank* 5.

3. Null Generalized Helices in $\mathbb{L}^5$

Definition 4 [8]**.** Assume that $\alpha \subset \mathbb{L}^5$ is a null generalized helix given by curvature functions k_1, k_2, k_3, k_4, k_5. Then the harmonic curvatures of α in $\mathbb{L}^5$ write-down as follows:

$$H_i = \begin{cases} -\dfrac{k_2}{k_1}, & i = 1, \\ \dfrac{H_1'}{k_3}, & i = 2, \\ -\dfrac{k_4}{k_5} H_2, & i = 3. \end{cases} \tag{3}$$

Definition 5 [4]**.** Let α be a time-like curve in $\mathbb{L}^5$ with $\alpha'(s) = V_1$. $X \in \chi(\mathbb{L}^5)$ being a constant unit vector field, if

$$\langle V_1, X \rangle = \cosh \varphi \text{ (constant)},$$

then α is called a *general helix* (inclined curves) in $\mathbb{L}^5$, φ is called *slope angle*, and the space $Sp\{X\}$ is called *slope axis*.

Definition 6 [8]**.** A null curve $\alpha : I \longrightarrow \mathbb{L}^5$ is said to be a generalized helix, if there exist a nonzero unit constant vector X such that $\langle \alpha'(t), X \rangle \neq 0$, is constant. Then $Sp\{X\}$ is called *slope axis* and for the Frenet frame $\{T, N, W_1, W_2, W_3\}$, we have

$$\begin{cases} \langle W_1, X \rangle = 0, \\ \langle N, X \rangle = H_1 \langle T, X \rangle, \\ \langle W_i, X \rangle = H_i \langle T, X \rangle, \quad 2 \leq i \leq 5. \end{cases} \tag{4}$$

Now, the Equation (1) can be given in terms of harmonic curvatures as follows.

Theorem 1. *Let* α *be a null Frenet curve of osculating order* 5 *in* $\mathbb{L}^5$. *Then*

$$\begin{cases} \nabla_T T = k_1 W_1, \\ \nabla_T N = -k_1 H_1 W_1 + \dfrac{H_1'}{H_2} W_2, \\ \nabla_T W_1 = k_1 H_1 T - k_1 N - \dfrac{H_3 k_5}{H_2} W_2 - \dfrac{H_2 k_4}{H_3} W_3, \\ \nabla_T W_2 = -\dfrac{H_1'}{H_2} T + \dfrac{H_3 k_5}{H_2} W_1, \\ \nabla_T W_3 = -\dfrac{H_2 k_4}{H_3} W_1, \end{cases}$$

where k_1, k_4, k_5 *are Frenet curvatures of* α; H_1, H_2, H_3 *are harmonic curvatures of* α; *and* ∇ *is the Levi-Civita connection of* $\mathbb{L}^5$.

Proof. By using Equations (1) and (3), we obtain the proof of the theorem.

□

Corollary 2. *If* $h = 0$ *and* $k_1 = 0$ *in* $\nabla_T T = hT + k_1 W_1$, *then* α *is a null geodesics in* $\mathbb{L}^5$.

Theorem 3. *Let* $\alpha : I \longrightarrow \mathbb{L}^5$ *be a null curve in* $\mathbb{L}^5$. *Then*

$$\begin{cases} \langle \nabla_T T, W_1 \rangle = -\dfrac{k_2}{H_1}, \\ \langle \nabla_T T, W_2 \rangle = \langle \nabla_T T, W_3 \rangle = \langle \nabla_T N, W_3 \rangle = \langle \nabla_T W_1, W_1 \rangle = 0, \\ \langle \nabla_T W_2, W_2 \rangle = \langle \nabla_T W_2, W_3 \rangle = \langle \nabla_T W_3, W_2 \rangle = \langle \nabla_T W_3, W_3 \rangle = 0, \\ \langle \nabla_T N, W_1 \rangle = -k_1 H_1, \\ \langle \nabla_T N, W_2 \rangle = \dfrac{H_1'}{H_2}, \\ \langle \nabla_T W_1, W_2 \rangle = -\dfrac{H_3 k_5}{H_2}, \\ \langle \nabla_T W_1, W_3 \rangle = -\dfrac{H_2 k_4}{H_3}, \\ \langle \nabla_T W_2, W_1 \rangle = -\langle \nabla_T W_1, W_2 \rangle, \\ \langle \nabla_T W_3, W_1 \rangle = -\langle \nabla_T W_1, W_3 \rangle, \end{cases}$$

where T and N are null vectors; $W_1, W_2,$ and W_3 are space-like vectors; $H_1, H_2,$ and H_3 are harmonic curvatures of α; ∇ is the Levi-Civita connection of $\mathbb{L}^5$; and k_1, k_2, k_4, k_5 are Frenet curvatures of α.

Proof. By using Equations (1), (2), and (3), we obtain the proof of the theorem. □

Theorem 4. *Let $\alpha : I \longrightarrow \mathbb{L}^5$ be a null curve in $\mathbb{L}^5$ and X be a nonzero unit constant vector field (time-like or space-like) of $\mathbb{L}^5$. Then*

$$\begin{cases} \text{(i) } \langle \nabla_T T, X \rangle = \langle \nabla_T W_1, X \rangle = \langle \nabla_T W_3, X \rangle = 0, \\ \text{(ii) } \langle \nabla_T N, X \rangle = H_1' \langle T, X \rangle, \\ \text{(iii) } \langle \nabla_T W_2, X \rangle = -\dfrac{H_1'}{H_2} \langle T, X \rangle, \end{cases}$$

where H_1 and H_2 are harmonic curvatures of α.

Proof. (i) $\langle \nabla_T T, X \rangle = \langle k_1 W_1, X \rangle = k_1 \langle W_1, X \rangle = 0, \ (\langle W_1, X \rangle = 0),$

$$\begin{aligned} \langle \nabla_T W_1, X \rangle &= \langle (-k_2 T - k_1 N + k_4 W_2 + k_5 W_3), X \rangle \\ &= -k_2 \langle T, X \rangle - k_1 \langle N, X \rangle + k_4 \langle W_2, X \rangle + k_5 \langle W_3, X \rangle \\ &= -k_2 \langle T, X \rangle - k_1 H_1 \langle T, X \rangle + k_4 \langle W_2, X \rangle + k_5 \langle W_3, X \rangle \\ &= -k_2 \langle T, X \rangle + k_2 \langle T, X \rangle + k_4 H_2 \langle T, X \rangle + k_5 H_3 \langle T, X \rangle \\ &= (k_4 H_2 - k_4 H_2) \langle T, X \rangle \\ &\Rightarrow \langle \nabla_T W_1, X \rangle = 0, \end{aligned}$$

$$\langle \nabla_T W_3, X \rangle = \langle -k_5 W_1, X \rangle = -k_5 \langle W_1, X \rangle = 0.$$

(ii) $$\begin{aligned} \langle \nabla_T N, X \rangle &= \langle (k_2 W_1 + k_3 W_2), X \rangle \\ &= k_2 \langle W_1, X \rangle + k_3 \langle W_2, X \rangle \\ &= k_3 \langle W_2, X \rangle \end{aligned}$$

$$= k_3 H_2 \langle T, X \rangle$$

$$\Rightarrow \langle \nabla_T N, X \rangle = H_1' \langle T, X \rangle.$$

(iii) $\langle \nabla_T W_2, X \rangle = \langle (-k_3 T - k_4 W_1), X \rangle$

$$= -k_3 \langle T, X \rangle - k_4 \langle W_1, X \rangle$$

$$= -k_3 \langle T, X \rangle$$

$$\Rightarrow \langle \nabla_T W_2, X \rangle = -\frac{H_1'}{H_2} \langle T, X \rangle.$$

□

Corollary 5 [8]. *α is a null helix in $\mathbb{L}^5$ $\Leftrightarrow$ $2H_1 + (H_2)^2 + (H_3)^2 =$ constant.*

Definition 7. A null curve $\alpha : I \longrightarrow \mathbb{L}^5$ is said to be a generalized helix, if there exist harmonic curvatures H_1, H_2, and H_3 such that

$$H_1' + H_2 H_2' + H_3 H_3' = 0.$$

Corollary 6. *$H_2' = -\frac{H_1'}{H_2}$ and $H_3' = 0$.*

Proof. From [8],

$$\langle W_i, X \rangle = H_i \langle T, X \rangle, \quad 2 \leq i \leq 5.$$

Thus

$$H_2' = \frac{\langle \nabla_T W_2, X \rangle}{\langle T, X \rangle} = \frac{-\frac{H_1'}{H_2} \langle T, X \rangle}{\langle T, X \rangle} = -\frac{H_1'}{H_2},$$

and

$$H_3' = \frac{\langle \nabla_T W_3, X \rangle}{\langle T, X \rangle} = 0.$$

□

4. Examples

Example 1. Let $\alpha : I \longrightarrow \mathbb{L}^5$ be the null curve defined by

$$\alpha(t) = (\sinh t,\ \cosh t,\ 1,\ 0,\ -t), \quad t \in R,$$

and $X = (0, 0, 0, 0, 1)$ a unit constant vector field in $\mathbb{L}^5$. The tangent vector of α is

$$T = \alpha'(t) = (\cosh t,\ \sinh t,\ 0,\ 0,\ -1),$$

and $\langle T, T\rangle = 0$, so α is a null curve in $\mathbb{L}^5$. Also, $\langle T, X\rangle = -1 =$ constant. Therefore, the curve α is a null helix.

Example 2. Let $\alpha : I \longrightarrow \mathbb{L}^5$ be the null curve defined by

$$\alpha(t) = (t,\ 0,\ \sin t,\ \cos t,\ 1), \quad t \in R,$$

and $X = (1, 0, 0, 0, 0)$ a unit constant vector field in $\mathbb{L}^5$. The tangent vector of α is

$$T = \alpha'(t) = (1,\ 0,\ \cos t,\ -\sin t,\ 0),$$

and $\langle T, T\rangle = 0$, so α is a null curve in $\mathbb{L}^5$. Also, $\langle T, X\rangle = -1 =$ constant. Therefore, the curve α is a null helix. Moreover, the frame $\{T, N, W_1, W_2, W_3\}$ is a distinguished Frenet frame along α, where from (2),

$$N = \frac{1}{2}(-1,\ 0,\ \cos t,\ -\sin t,\ 0),$$

$$W_1 = (0,\ 0,\ \sin t,\ \cos t,\ 0),$$

$$W_2 = (0,\ 1,\ 0,\ 0,\ 0),$$

$$W_3 = (0,\ 0,\ 0,\ 0,\ 1).$$

Thus, from (4), we can find the following results:

$$H_1 = -\frac{1}{2}, \quad H_2 = H_3 = 0.$$

Example 3. Let

$$\alpha(t) = \left(\sqrt{3}\sinh t, \sqrt{3}\cosh t, t, \cos t, \sin t\right), \quad t \in R,$$

$$V_1 = \alpha'(t) = \left(\sqrt{3}\cosh t, \sqrt{3}\sinh t, 1, -\sin t, \cos t\right),$$

where $\langle \alpha'(t), \alpha'(t) \rangle = -1$, which shows $\alpha(s)$ is time-like curve and $X = (1, 0, 0, 0, 0)$ a unit constant vector field in $\mathbb{L}^5$. Then,

$$\langle V_1, X \rangle = -\sqrt{3}\cosh t = \text{constant}.$$

Thus $\alpha(t)$ is a general helix in $\mathbb{L}^5$.

Example 4. Let $\alpha : I \longrightarrow \mathbb{L}^5$ be the null curve defined by

$$\alpha(t) = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, 0, \cos t, \sin t), \quad t \in R.$$

The tangent vector of α is

$$T = \alpha'(t) = \frac{1}{\sqrt{2}}(\cosh t, \sinh t, 0, -\sin t, \cos t),$$

and $\langle T, T \rangle = 0$, so α is a null curve in $\mathbb{L}^5$. Moreover,

$$\nabla_T T = \frac{1}{\sqrt{2}}(\sinh t, \cosh t, 0, -\cos t, -\sin t),$$

and

$$\langle \nabla_T T, \nabla_T T \rangle = 1 > 0,$$

$\nabla_T T$ is a space-like vector field, so we can take $\nabla_T T = W_1$, which implies that $h = 0$ and $k_1 = 1$ in the first equation of (1). Thus, $h = 0$ implies that t is the distinguished parameter for α and by Corollary 2, α is a non-null geodesic in $\mathbb{L}^5$. By taking the derivative of W_1 with respect to T, we have

$$\nabla_T W_1 = \frac{1}{\sqrt{2}}(\cosh t,\ \sinh t,\ 0,\ \sin t,\ -\cos t).$$

Choosing

$$W_2 = \frac{1}{\sqrt{2}}(\sinh t,\ \cosh t,\ 0,\ \cos t,\ \sin t),$$

and taking the derivative with respect to T, we have

$$\nabla_T W_2 = \frac{1}{\sqrt{2}}(\cosh t,\ \sinh t,\ 0,\ -\sin t,\ \cos t) = T.$$

This implies that $k_3 = -1,\ k_4 = 0$ from $\nabla_T W_2 = -k_3 T - k_4 W_1$ and we obtain

$$N = \frac{1}{\sqrt{2}}(-\cosh t,\ -\sinh t,\ 0,\ -\sin t,\ \cos t).$$

By taking the derivative of N with respect to T, we have

$$\nabla_T N = \frac{1}{\sqrt{2}}(-\sinh t,\ -\cosh t,\ 0,\ -\cos t,\ -\sin t) = -W_2.$$

This implies that $k_2 = 0$ in the second equation of (1). Choosing

$$W_3 = \frac{1}{\sqrt{2}}(-\cosh t,\ -\sinh t,\ 0,\ \sin t,\ -\cos t),$$

and taking the derivative with respect to T, we have

$$\nabla_T W_3 = \frac{1}{\sqrt{2}}(-\sinh t,\ -\cosh t,\ 0,\ \cos t,\ \sin t) = -W_1.$$

This implies that $k_5 = 1$ in the fourth equation of (1). Thus, the harmonic curvatures of α are

$$H_1 = H_2 = H_3 = 0.$$

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